Perhaps a Matter of Imagination: TPCK in Mathematics Education

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"Imagination is more important than knowledge. For knowledge is limited to all we now know and understand, while imagination embraces the entire world, and all there ever will be to know and understand." -- Albert Einstein

Introduction

Albert Einstein was fond of discussing the need to mix imagination and knowledge in understanding the world of mathematics. His opinions on the topic often surfaced during his public conversations and various presentations, especially those after his 1921 Nobel Prize in Physics. When you examine his many different quotes and thoughts related to mathematics, education, and life, you get a sense of the dynamic and engaging educational environment that we are still striving for today in our nation's schools. His frequent comments on independent thinking, questioning, and connecting mathematics to the real world all seem relevant today when considering what teaching and learning should look like in today's mathematics classroom (for a good source on Einstein see the Priwer and Phillips book "Everything Einstein", 2003).

But the world has changed considerably since Albert Einstein died in 1955, and especially the technology of our world and the tools that we now have available to us as educators in the mathematics classroom. A mere 50 years has seen us move from slide rulers to graphing calculators and from thick books of computational tables to the impressive computing power of laptop computers. Throw into the mix the educational potential of the Internet and even an impressive thinker like Albert Einstein might be challenged to imagine just how it all comes together for an effective learning environment in a mathematics classroom. Yet, for the pre-service and inservice mathematics teachers of today, there is an expectation that upon graduation from our teacher education programs they will be able to design creative and effective learning activities that take full advantage of educational technology. We as the leaders of those teacher education programs are perhaps challenged most of all, as we try to systematically help these pre-service and inservice teachers imagine how such technologies might be used in developing lessons that engage, excite, and enhance the learning of an increasingly diverse set of students.

The effective use of technology in the mathematics classroom has long been a topic for consideration by mathematics educators. One simply needs to read some of the many discussions on when calculators should be used with students in the learning of mathematics to get a sense of this historical and sometimes conflicting dialogue related to the use of technology in the learning of mathematics (Ball & Stacey, 2005; Heid & Edwards, 2001). Some questions have surfaced time and again in these discussions. Questions such as should students who don't know the fundamentals of basic arithmetic operations be allowed to use a calculator? How about if those same students are doing a real-life application activity that requires a great deal of repetitive arithmetic, should calculators be allowed then? The answer to many of these technology-related questions would probably be given by most mathematics educators today as "it depends." It essentially depends upon how the content, pedagogy, and technology might best intersect within the learning of a specific mathematical topic. It also depends on the individual learners themselves, with the diverse needs and background experiences that they each bring to the mathematics classroom.

When considering the overall use of technology in mathematics education, like other fields, it would seem that we have suffered from too much of a focus on "what" technologies to use and too little imaginative thinking on "how" these technologies might be used to support teaching and learning (Heid, 2005; Sinclair & Crespo, 2006; Horwitz & Tinker, 2005). Most of all, we have not had a framework or programmatic theme upon which to build that imaginative thinking and technology use (Hofer, 2005). The current emphasis on technological pedagogical content knowledge (TPCK) as represented in this monograph and foundational articles (such as Mishra & Koehler, 2003; In Press) is quite timely for those of us with a particular interest in preparing teachers of mathematics. How and when technology should be used within our field has never been an easy question for us, or one that is taken lightly.

For mathematics educators, defining the most effective uses of technology in the teaching of mathematics can certainly be described as a “wicked problem”, as represented by Koehler and Mishra in the first chapter of this monograph. A number of challenging instructional questions are associated within this wicked problem, such as: When should teachers incorporate calculators when teaching arithmetic? How should teachers incorporate the powerful new symbolic processing programs within basic algebra instruction? Should teachers allow student use of the many new online homework assistance websites for mathematics? Such instructional questions illustrate how the problem of effective technology integration into the teaching of mathematics fits the parameters of a wicked problem that is indeed ill-structured, complex, and exists within a dynamic context of interdependent variables (Rittel & Webber, 1973). Instructional questions within this context continue to lead to contextually and politically charged discussions among mathematics educators (Ball and Stacey, 2005).

Part of our current challenge in understanding how and when we use technology effectively has been that the mathematics content itself is a rapidly moving “target.” Look at most new books on mathematics that reside on the shelves in one of your local mega-bookstores and it is relatively easy to see that technology has had a considerable impact on the content of the mathematics discipline itself. Consider the concept of "fractals". Fractals are essentially a representation of objects that have "fractional" geometric dimensions. Moving beyond the traditional geometry of points, lines and planes, fractal geometry instead uses computer-like "algorithms" that replicate relatively intricate geometric patterns (Lesmoir-Gordon, 2001; Peterson, 1998; Turner, Blackledge, Andrews, 1998). A representative and relatively famous fractal is the Sierpinski Triangle, which was named after the Polish mathematician Waclaw Sierpinski, who investigated some of its interesting properties. The triangle illustrates a common fractal property of “self-similarity”, where the larger triangle divides into smaller similar triangles, that then divide into even smaller triangles, and the pattern continues. A computer generated Sierpinski Triangle is shown below.

--------- Insert Image of the Sierpinski Triangle about here ----------

Fractal algorithms like Sierpinski’s Triangle help us to define some real world phenomena in ways that we never could before using fractional dimensions. For example, clouds, coastlines, plant growth, lightening and even blood vessels, are now routinely defined with fractals and systematically investigated with computer technology, due to the iterative nature of these phenomena (Falconer, 2003). Striving for accuracy in defining or measuring various geographic features like coastlines or rivers is a particularly good example of how fractal mathematics and computer technology can be useful today. When someone (such as a map maker) attempts to measure a coastline, the often jagged line of where water meets shore can actually be measured from many perspectives, ranging from the height of a satellite to the very close viewpoint of a person kneeling on the shore. Using fractal algorithms that make allowances for such considerably different scales, computers can now provide a much more accurate and consistent measurement of coastlines. Another good example of where fractals and computer technology can be useful is for predicting the flooding of rivers with certain geographic patterns. In some rivers, a branching pattern is formed as various tributary streams flow into the larger river. This "branching pattern" can significantly impact water flow and is at times associated with a potential for devastating flooding. Fractal algorithms and computers have greatly increased our understanding and possible prediction of flooding in such river systems. A public domain NASA image from the SeaWiFS satellite of a river system in Norway illustrates this fractal branching pattern of some rivers (visible within snow fields and taken on June 6, 2000).

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Fractals have become a mathematical topic that probably should now be included in every high school curriculum, if for no other reason than that they have considerable real life applications and connections. Yet the mathematics of fractals can be confusing. For students to understand fractional geometric dimensions they need to understand "fractional exponents," a topic that relies heavily on understanding both fractions and exponents. How and when a teacher might facilitate the learning of fractals, using computer technology to its full advantage, is probably a good example of our need to help mathematics teachers develop strong backgrounds in TPCK.

Fractals are just one illustration of the many important topics in mathematics that have evolved and changed due to technology. Topics such as statistics, graphing, coordinate geometry, matrices, probability, combinatorics and many other mathematics-related topics are in an ongoing state of change and evolution (Heid, 2005). For example, technology has enabled a wide array of advanced and multivariate statistical methods to recently evolve for analyzing complex data sets (Mertler and Vanatta, 2005). Mathematics is not the only discipline evolving and as one might imagine, other disciplines are being impacted by technology as well. Schmidt and Gurbo's discussion of K-6 literacy education in Chapter 3 is a good illustration, where hypermedia technology is changing how we fundamentally read and search for information.

Recent national studies in mathematics education (and in related fields such as science) have left little doubt that pedagogy and content must be interwoven by teachers to achieve dynamic and effective educational environments within this evolving context for mathematics (Kim, 2003; Martin, 2004; National Science Board, 2003). As technology's role in this instructional mix continues to expand and evolve, it would seem increasingly important that teachers be adept at deciding where technology fits in such mathematics instruction. If technology is left out, teachers may well be missing an important opportunity for aiding their students' understanding; or perhaps worse for some topics, teachers may even misrepresent mathematical concepts that are tied closely to computers (such as fractals). On the other side, if teachers use technology inappropriately or too freely (such as having students rely too heavily on calculators in the learning of basic arithmetic), they run the risk of deepening student misconceptions and expanding bad habits. Thus, addressing TPCK carefully within teacher preparation may well be the difference between students having an effective or ineffective mathematics teacher.

What Really is the Study of Mathematics?

For us to consider the TPCK needed by teachers in the effective instruction of mathematics, one might first review a bit about the discipline of mathematics itself. What really is the study of mathematics? What are we trying to learn when we study it in our classrooms? Such questions are actually more difficult than they may seem to be at first. Ask many elementary level students and their teachers to define mathematics and it will often be defined as a study of arithmetic and simple geometry. Middle level and high school level students and teachers might also mention variables, and perhaps various mathematical topics such as algebra, geometry, trigonometry, and calculus. However, you can definitely expect a long pause before any of these individuals answer your question, because it is not a simple question to answer.

In many ways, defining mathematics by its topics (such as involving arithmetic, geometry, algebra, etc.) is much like defining music as a mere sequence of sounds. The "human" or creative element is missing from such a definition, and that human element helps give mathematics, like music, its own interest and beauty. A superficial definition of mathematics can miss its spirit and depth, ignoring the joy, excitement, and even utility that may be found in its study. For those who take the time to look more deeply, mathematics often represents a rich and dynamic excursion into trying to know and control our world through its patterns. It is a discipline that helped mankind build pyramids, navigate oceans, and send rockets into space. It is a discipline in which the imagination and logic of the human mind strive to structure the reality of our existence.

An excellent contemporary discussion and definition of mathematics is the discussion presented by Steen, who suggests:

> *Mathematics involves observing, representing and investigating patterns and relationships in social and physical phenomena and between mathematical objects themselves: Mathematics is often defined as the science of space and number … [but] a more apt definition is that mathematics is the science of patterns. The mathematician seeks patterns in number, in space, in science, in computers, and in imagination. Mathematical theories explain the relations among patterns … Applications of mathematics use these patterns to 'explain' and predict natural phenomena … (1988, page 616.)*

To see and teach the discipline of mathematics in this exciting way, especially in the complex world of today, a teacher must have the knowledge and overlapping skills represented by TPCK. For teachers of mathematics, it would seem that adequate knowledge in each of the three areas of technology, pedagogy, and content is relatively critical to a teacher's effectiveness in today's mathematics classroom. When considering the content of mathematics, the teacher must know enough of the content of mathematics to be able to appreciate and represent the depth of this important discipline. Secondly, the teacher must comprehend enough of the pedagogy of teaching mathematics to help students systematically build toward an understanding of mathematics similar to the teacher's own understanding. And finally, since much of the mathematics of today is intricately interwoven with technology, the teacher must be able to understand and use the technology of mathematics in its instruction. But as stated by Mishra, TPCK is much more than the individual three components and "represents a thoughtful interweaving of all three key sources of knowledge - technology, pedagogy, and content" (In Press, p. 14).

Mathematics Teaching and TPCK

Consider the rapid growth of technology in our society, the variety of pedagogical strategies available to teachers, and the evolving nature of mathematics itself. Anyone reflecting upon how to develop TPCK in mathematics teachers quickly discards the strategy of trying to have teachers gain an experience in all of the potential combinations where technology, pedagogy and content come together in the mathematics classroom. Such a listing would no doubt be more of an exercise in understanding the mathematical concept of infinity than it would be a strategy conducive to strong teacher preparation. Instead, since it would appear that we cannot possibly provide teachers with all of the potential TPCK that they might use within a mathematics classroom, we must instead help them to imagine "possibilities" and develop an open mind for using a variety of approaches and strategies with their students.

We must also help teachers to make sure that the mathematics learned in the classroom is truly mathematics, and not some limited or “watered-down” version of mathematics in the pursuit of a more engaging classroom. Current recommendations, such as those presented within the American Educational Research Association’s panel report, “Studying Teacher Education,” emphasize the importance of a strong sequence of college mathematics coursework for teachers at all levels engaged in the teaching of mathematics (AERA, 2005). This document references correlations between the college coursework of prospective teachers and the general mathematics achievement of their students. Strong content preparation is clearly important for mathematics teachers.

Yet content in mathematics is a much broader realm than many people, including most novice teachers, might realize. This realm is also expanding rapidly. Consider what has been called “Discrete Mathematics.” This field of mathematics involves the study of objects or ideas that can be divided into discrete (separate or discontinous) parts (Rosen, 1999). Where continuous mathematics is typically dealing with measurement, discontinous mathematics is more focused on counting. For a simple definition, some authors have also described discrete mathematics as "the mathematics most relevant to computing" (Gardiner, 1991). There are actually some problems in discrete mathematics that seem simple at first glance, but really need significant computing power to solve. Problems that have been called "knapsack problems" are good examples. Consider the classroom problem presented by Caldwell and Masat (1991), which has various real life connections.

> "You are taking a two-week hike and will be backpacking everything you need. You have made a list of eight possible items to take of a total of 77 pounds, and your list has each item's weight and its value to you rated from 1 to 5, with 5 being the highest. If you carry only 30 pounds, what should you take along to get the highest number of value points? (p. 228).

This problem seems simple at first and you might make some initial headway using a spreadsheet and ordering the items by weight, or by value points, or even by a ratio of weight per value point for each item. However, you eventually find that for the lesser items, you also need to try various combinations. To try all possible combinations you will be trying 2^8 or 256 combinations, and even with access to a spreadsheet with sorting capabilities, all of a sudden writing a computer program to try all possible combinations doesn't sound so bad (and is actually probably a better strategy). Such real life "knapsack" problems are common in today's world, such as when NASA tries to decide what experiments (each with relative value and weight) might be included on a space mission. Teaching such new computer-based mathematics relies directly on teachers with strong TPCK backgrounds.

Carefully chosen examples, such as the discrete mathematics backpack problem can go a long way toward providing teachers with strategies for approaching the instruction of various mathematics topics. As Shulman mentions, building pedagogical content knowledge (PCK) is basically related to providing "the most powerful analogies, illustrations, examples, explanations, and demonstrations" to make the subject accessible and comprehensible (1986). Such foundational experiences can have a significant relationship to later teacher attitudes and classroom strategies (Shulman, 1987; Frykholm and Glasson, 2005). As disciplines get more complex and abstract for teachers, such carefully chosen examples will become even more important (Gates, 2004). Providing examples that connect to technology will also be important for addressing the changing nature of mathematics due to technology. Beyond new areas such as discrete mathematics, even more foundational mathematical areas such as algebra are being impacted by their relationship to technology (Heid & Edwards, 2001; Hegedus & Kaput, 2004). For example, some authors suggest that a renewed focus on algebraic thinking is due to the significant role that algebra now plays in technology-related careers and the usefulness of understanding algebra to help comprehend the symbolic nature of how computers process information, while better ensuring an equitable education for all students (Checkley, 2006; Moses, 2000; Moses, 1994).

In the mathematics classrooms of today's schools, teachers may often find themselves helping their students to "imagine" the relationships and patterns in numbers, space, science, computers, and even in thinking itself. The National Council of Teachers of Mathematics (NCTM) has embraced this dynamic approach to mathematics instruction in its *Principles and Standards for School Mathematics* published by NCTM in 2000. The instructional principles represented in this document provide a "vision" for equity in student expectations, a coherent curriculum, dynamic teaching, constructivist learning and formative assessment. In its vision of mathematics instruction, the document describes two strands of standards, that of content and that of process. Content standards address the important aspects of mathematical content that should be learned and include numbers and operations, algebra, geometry, measurement, and data analysis/probability. Process standards cover the ways and strategies in which mathematics might be used and include problem solving, reasoning/proof, communication, connections, and representation. For further refinement of the mathematics content to be taught at the elementary and middle school levels, the NCTM recently published the 2006 document *"Curriculum Focal Points for PreKindergarten through Grade 8 Mathematics: A Quest for Coherence"* which strives to limit the number of mathematics topics taught at each of the PreK through 8 grade levels so that teachers can cover this content more efficiently and in more depth. Thus, when considering the pedagogy and content aspects of TPCK needed for encouraging effective mathematics instruction, it is important to remember that the NCTM standards and curriculum focal points represent a strong foundation upon which to build.

The NCTM has clearly started to address the interaction of technology more directly within the general context of TPCK. In 2003, the NCTM published a position paper on technology that gives significant insight into how technology should be used in the mathematics classroom and the potential role of TPCK. This position statement makes it clear that technology is an essential part of a current mathematics curriculum and states that "using the tools of technology to work within interesting problem contexts can facilitate a student's achievement of a variety of higher-order learning outcomes, such as reflection, reasoning, problem posing, problem solving and decision making" (pg. 1). The position paper also makes a series of five recommendations that support this premise and recommends that all levels and all courses should include strong and thoughtful uses of technology. One of the most interesting recommendations in this document is the fourth recommendation, which states: "Programs of preservice teacher preparation and in-service professional development should strive to instill dispositions of openness to experimentation, with ever-evolving technological tools and their pervasive impact on mathematics education" (pg. 2). Such a recommendation reflects a subtle but significant change for mathematics education and recognizes that it is to some degree, the role of the individual mathematics teacher to "imagine" how new technologies might be used in the mathematics classroom and to involve students in experimentation with these important tools in the context of their learning.

Some of the most recent and carefully undertaken research in mathematics education today appears to be consistent with a systematic march toward TPCK as a unifying theme in which we can conceptualize good mathematics teaching with technology. A good example is the NCTM 2005 Yearbook. It is entitled "Technology Supported Mathematics Learning Environments" and includes 23 excellent articles that carefully suggest technology-supported activities for the mathematics classroom, while also carefully mapping these strategies to both content and pedagogy. Many of the titles themselves suggest such connections. For example, an article by Vincent, entitled "Interactive Geometry Software and Mechanical Linkages: Scaffolding Students' Deductive Reasoning" does an excellent job of linking specific technologies to particular geometry concepts, using a strong research-based pedagogy (p. 93-112). Other articles also reflect a more collaborative approach in defining such TPCK-based activities, such as an excellent article by eight university researchers and public school teachers (Reece, et.al,) which is entitled "Engaging students in authentic mathematics activities through calculators and small robots" (p. 319-328). Similarly, many of the more established professional journals in mathematics education are also showing a new context for TPCK, such as the May

2006 issue of Teaching Children Mathematics, which had a lead article by Sinclar and Crespo, that was entitled “Learning Mathematics in Dynamic Computer Environments.”

How we approach the instruction of mathematics in our schools and our corresponding preparation of mathematics teachers is evolving due to technology. The future of technology use in the mathematics classroom is typically seen as an exciting one by mathematics educators. Consider the recent work of Heid (2005), who interviewed 22 leading educators in technology and mathematics education and identified areas that seemed to be the most promising in technology use for the mathematics classroom. These technology applications included various innovations, such as dynamic computation tools, or software that facilitates interactive computation; microworlds, which allow focused experimentation of various mathematical relationships; intelligent tutors, which facilitate flexible instruction of mathematical content; computer algebra systems, which facilitate symbolic manipulation of algebraic symbols; handheld devices, which permit convenient technology access; web-based instruction, which allows systematic, cost-effective instruction; and interactive learning communities, that can facilitate educational collaboration in classrooms across many different states and even nations.

Beyond instruction, technology can also play an important administrative role for mathematics teachers and their schools. Education Week recently showcased the role of computer technology in helping teachers with administrative tasks in their classroom such as recording and tracking criterion referenced tests, class assignments and attendance, as well as connecting that information to suggestions for lessons (Hoff, 2006; Zehr, 2006). Districts are steadily becoming more “data wise” and helping their teachers to store, access and interpret information within an electronic format to aid them in making pedagogical decisions in the classroom (Borja, 2006). Some authors are even declaring Internet access within most classrooms today as generally “ubiquitous” while also suggesting that more needs to be done with teacher professional development to help teachers better understand how to use such pervasive computer access to improve instructional decisions at the classroom level (Swanson, 2006).

As technology is becoming ever more pervasive in our society it is also being more formally recognized as a critical tool for facilitating both the effective doing and learning of mathematics. The Association of Mathematics Teacher Educators (AMTE) succinctly reinforced this critical importance of technology in the teaching and learning of mathematics in their recent position paper, when they stated that “AMTE recognizes that technology has become an essential tool for doing mathematics in today’s world, and thus that it is essential for the teaching and learning of mathematics” (pg 1, 2006). The AMTE position paper is a strong endorsement of the idea that one cannot truly be an effective teacher of mathematics in today’s classroom unless one has “the knowledge and experiences needed to incorporate technology.” The two-page AMTE position paper also reinforces a mathematics teacher’s need to be both flexible and creative in their application of technology.

Qualities of a Mathematics Teacher with TPCK

A teacher who has a strong background in TPCK offers her students a considerable advantage in the learning of mathematics. Consider the advantages that such a teacher might have in introducing algebra into a middle school classroom on the first day of an algebra class. I had the chance to witness this advantage first hand while watching a middle school teacher with 20 years of classroom experience and strong TPCK skills. She first approached the introduction of algebra by showing how it generalizes arithmetic by giving examples of specific triangles and finding their areas. This was first illustrated by having students use an electronic geoboard (an applet where virtual rubber bands are stretched on virtual pegs) available at the National Library of Virtual Manipulatives site (http://nlvm.usu.edu/en/nav/vlibrary.html). The teacher then decided to use a spreadsheet on her display device to further illustrate the area formula for a triangle by showing various examples and reinforcing the mathematical relationship between the triangle's base, height, and area; as well identifying how algebra within the spreadsheet helped students to more efficiently compute the areas of any triangle. The teacher then further illustrated algebra in real life by having students visit various websites and investigate several careers where algebra played a significant role. Much of the teacher's lesson was relatively spontaneous and relied on her ability to "imagine" the next step in the instructional process. It was easy to see by the attentive looks of the students and their nodding heads that they better understood the "power and utility" of algebra and that they were now ready to start their year of algebra study.

Could this introduction of algebra have been done without the computer, or facilitated by a teacher with little TPCK? Perhaps, but considering that several aspects of this lesson were initiated in response to student questions (such as the possible career connections to algebra) it would seem doubtful that a teacher without strong TPCK could have assembled such an effective lesson so quickly. A teacher without TPCK would not have decided to use a virtual geoboard for their initial illustration and even with regular geoboards available (obviously a worthy tool in its own right), the focus of the lesson was not on the triangles, but rather on the generalizing of arithmetic represented by algebra. The visible connection between the individual triangles and the area formula for triangles was a further advantage offered by a computer spreadsheet in this context, and the teacher knew that this was an excellent opportunity to take advantage of that tool. Using a spreadsheet also showcased the power of algebra within a real life computational environment represented by the spreadsheet itself. The websites associated with the algebra related careers (perhaps her most spontaneous part of the lesson) established a further real life context for students and helped to directly answer the age old classroom question, "why do we need to know this?" before it ever surfaced. A mathematics teacher without the aid of technology, or even one with technology access but little TPCK would be hard pressed to do so much in so little time with this introductory lesson. Algebra by its very nature is abstract and many an algebra student has spent their first few weeks, and some their first full year of algebra instruction, without really being able to define algebra (Noddings, 2000; Steen, 1992). These particular students now had an excellent working definition for algebra, as well as several examples of its use in real life, before they commenced study of this critically important body of mathematical content.

This talented teacher's ability to imagine potential applications of computer technology and then "weigh" the relative benefit of these various instructional options within the context of a specific mathematical topic may be what TPCK for mathematics teachers is all about in today's classroom. Ever since the first "A Nation at Risk" report was published (National Commission on Excellence in Education, 1983) reform in mathematics education has been steadily striving to make the learning of mathematics more relevant, more engaging, and in many ways, more imaginative. Imagining technology connections, determining the benefit of related instructional strategies, and putting it all together for an effective mathematics lesson is really a growing responsibility for the mathematics teacher of today. As suggested by the NCTM position statement on technology: "Technologies are essential tools within a balanced mathematics program. Teachers must be prepared to serve as knowledgeable decision makers in determining when and how their students can use these tools most effectively." (NCTM, 2003).

Imagination is more than a component of good mathematics instruction; it is really at the core of mathematics itself. Like the initial quote from Albert Einstein at the beginning of the article, few mathematicians and their colleagues in mathematics education would argue against the importance of using ones imagination in the study of mathematics. In fact, in some mathematical topics, such as with complex (or imaginary) numbers, the human imagination plays a critical role in understanding the content. Some works associated with mathematics are even known for their use of imagination, such as the paintings of M.C. Escher and his engaging use of tessellations. Interestingly, Escher was rarely successful at traditional mathematics learning and once said in an interview, "I never got a pass mark in math ... just imagine -- mathematicians now use my prints to illustrate their books."

Reaching all students in the mathematics classroom, such as talented artists like Escher, or students with different learning styles, genders, races and backgrounds, may well become steadily more linked to effective technology use. The computers' flexibility with instructional scaffolding, alternative representations, screen displays, audio languages, assessment and teacher feedback makes reaching a wider range of students increasingly more possible as computers become more pervasive and ubiquitous (Horwitz & Tinker, 2005). Such innovation and classroom flexibility has also been identified by the National Council of Teachers of Mathematics as a key to narrowing mathematics achievement gaps (NCTM, 2005). Yet, the ability of a teacher to take advantage of such technology-based instructional flexibility in reaching all students may well depend significantly upon that teacher's knowledge of which technologies and pedagogies to employ for which content.

It would make sense then, that if a teacher education program wants to build strong TPCK within its mathematics education students (both undergraduate and graduate), such a program might need to consider how to encourage these future and current teachers to approach mathematics instruction more imaginatively and creatively and to strive for lessons that engage, excite and educate the students within their own classrooms. But first, how would we recognize mathematics teachers who have successfully achieved such skills? In other words, what qualities might be observable in mathematics educators who are strong in their TPCK? What qualities were observable in the teacher mentioned earlier, who represented algebra so well using technology? Based on the previous discussion, it makes sense that mathematics teachers with strong backgrounds in TPCK would most likely demonstrate the following characteristics.

- Mathematics teachers with strong backgrounds in TPCK would probably have a relative openness to experimentation with the ever-evolving technological tools available to them in the mathematics classroom. In other words, they will "try" new technology-based lessons with their students on a regular and sometimes spontaneous basis, confident that if done thoughtfully and interactively, their students can learn something of value each time they attempt something new.

- Such mathematics teachers would also probably strive to be consistently "on-task" for the mathematical topic or content being taught. Teachers with strong pedagogical content knowledge, regardless of technology, would stay relatively focused on the content being discussed or explained during their lessons. In other words, teachers with strong TPCK are effective at focusing on the mathematics concepts, while still taking advantage of the instructional opportunities offered by technology.

• Mathematics teachers with strong TPCK backgrounds would also approach their mathematics instruction with clear and systematic pedagogical strategies in mind. In other words, these teachers would strive to know "where" their students are conceptually, "what" they need to do to achieve the next step in an instructional process, and "how" they generally want their students to proceed through careful sequences of classroom interactions and tasks.

• Mathematics teachers with strong TPCK would try to make periodic connections for their students as to "why" a particular technology is useful for instructing a particular mathematics topic. In other words, these teachers would consistently offer explanations to their students on what they are doing with the technology, why a specific tool is appropriate for a particular mathematical situation and perhaps even how a selected technology fundamentally works. Such explanations of some classroom technologies (such as computer-based lab equipment, graphing calculators, or wireless Internet access) can also have an added benefit of contributing to a student's understanding of the sciences related to the technology, as suggested by McCrory in the science chapter of this monograph (Chapter 9).

• Strong TPCK teachers would also characteristically embrace the administrative capabilities of technology to help guide their mathematics instruction using student assessment data such as criterion-referenced tests. Such assessment data can help a teacher identify gaps in student understanding which might form the rationale for switching instructional strategies or taking a different pedagogical approach with some or all of the students. This "data-driven" decision-making process can help a teacher select lessons that more directly address where students are in their current understanding of a topic. It also models for students how the computer might aid in the management of information.

- Perhaps most of all, mathematics teachers with strong TPCK would also do their best to be caring teachers who are comfortable and optimistic for change. Consistent with a definition of mathematics as the dynamic discipline that it is in today's world, a teacher with a rich TPCK background would expect change, not only in the technologies available to them in the classroom, but quite likely in the content of the mathematics that they should be teaching. They must be caring instructional leaders that are welcoming to all students as they enter this changing and evolving world of mathematics.

Mathematics Teacher Education Programs and TPCK

How do we develop dynamic and caring teachers with TPCK like the one described above? Are they "super teachers" who are born to excel in a technology-enabled classroom, or are they simply teachers who have been prepared to excel with the necessary background experiences? Most teacher educators would no doubt suggest that an effective teacher education program can indeed have a significant impact on later teacher and student achievement, or why have such a program in the first place? It would make sense that if we are to develop TPCK within the preservice and inservice mathematics teachers of a teacher preparation program, then like most programmatic objectives, we must consciously address this goal across our program operations (Education Commission of the States, 2003). What then might such a program look like? What might be its observable features? Consistent with Einstein's endorsement of imagination, let's take a few last moments to "imagine" how such a teacher preparation program in mathematics education might approach building TPCK within their preservice and inservice teachers.

Since software, hardware, and computer applications are changing so rapidly, it is probably quite impossible for a teacher preparation program to directly teach its students all the individual applications and possibilities that might arise in their use of technology within the context of content and pedagogy. Instead, such programs can essentially only hope to foster a "disposition" for the use of such technologies in a flexible, experimental, and thoughtful way. This might seem to be particularly true as we move into educational environments that become ever more pervasive in computing technologies (Kaput, 2000).

In many ways, to be successful, an effective teacher preparation program must be relatively dynamic, reflective and transformative (Thompson & Zeuli, 1999). Some foundational glimpses of what is important in such teacher preparation environments for mathematics appears to be steadily evolving from research and is becoming more directly related to how students at all levels learn effectively. In the National Research Council's 2005 report "How Students Learn," a total of 179 out of the 600 pages are dedicated to the learning of mathematics. Within this extensive discussion, Fuson, Kalchman, and Bransford (pgs. 217-256), reinforce that there are three important principles for teachers to follow in helping provide a foundation for the learning of mathematics. These principles include: 1) teachers must engage student prior understandings; 2) teachers must help students build a deep foundation of factual knowledge, give students a conceptual framework, and help them to organize knowledge; and 3) teachers need to help students take a metacognitive approach in taking control of their own learning. For mathematics education, perhaps more than in many other disciplines, the newest studies are reinforcing that such learning principles must also be strongly grounded in content preparation, as well as practiced in multiple contexts and field experiences (AERA, 2005; U.S. Department of Education, 2005). Such guiding principles related to how students need to learn, when grounded in strong content preparation and technology use, would seem to provide an excellent framework upon which to help mathematics education programs organize instruction and instill strong TPCK in their graduates.

In today's environment of teacher preparation, programs that successfully address the TPCK of their preservice and inservice teachers would also need to ensure that such teachers are prepared for the many culturally diverse settings in which they might teach. Various studies have shown that there is a definite obstacle to minority students' mathematics achievement, possibly even more so in urban settings, when there is a mismatch between teachers' personal knowledge of students' cultures and the actual cultural environments of their schools (Seilar, Tobin, & Sokolic, 2001). For example, urban teachers sometimes find that they are better able to build the mathematical understanding of their students when they use urban illustrations for challenging mathematical topics, such as explaining network optimization by use of a traffic flow example. When teachers are not aware of such potential cultural contexts, they miss out on a very powerful way of connecting with their students (Kaser et. al., 1999; Ladson-Billings, 1995).

As teacher preparation programs strive to more systematically embrace TPCK as a foundational goal for preparing mathematics teachers and for reaching all students regardless of their culture, then TPCK may well become a useful "organizational construct" for helping prioritize technology-related experiences. When a program's leaders or instructors consider a particular technology-related activity for potential inclusion in a mathematics education program, then experiences that are more directly compatible with TPCK may be worthy of a higher priority. For example, a spreadsheet activity that explores and addresses student misconceptions of algebraic functions would seem to be a relatively important experience, while using Photoshop to modify images for illustrating the cover a math journal might seem less important (although perhaps worthy for another context).

Assuming that TPCK indeed represents a key construct for a strong mathematics education program in today's teacher preparation environment, it would make sense that successful programs might exhibit the following characteristics as they strive to develop a strong TPCK foundation in their students.

1) A successful program would most likely need to encourage an "imaginative openness" for classroom experimentation in using technologies for learning mathematical content. This encouragement would no doubt be most observable in how the professors or instructors themselves experiment with technology and various learning strategies in pursuit of their own learning goals within the program's coursework. Such experimentation is consistent with the "disposition" recommendation as mentioned in the NCTM position statement on technology.

2) A successful program would probably not overly separate technology, content and pedagogy across the coursework of teachers. In other words, the required courses within the mathematics department would include strong technology use and an effective pedagogy of presentation; while correspondingly, the methods and core education courses would also strive to periodically connect to a strong mathematical context.

3) It would seem that a teacher preparation program must carefully select the TPCK-related examples or problems that would be included in a methods class or other program coursework. Instructional opportunities represented in this way may well form one of the more important strategies for building TPCK within teachers when considering that modeling can be so powerful in teacher professional development (Education Commission of the States, 2003).

4) As suggested by Mishra (In Press), TPCK itself may well represent an important framework for restructuring the professional development experiences for teachers. This framework seems particularly important within the context of mathematics, as the discipline itself continues to evolve and expand related to technology.

5) TPCK experiences in which a teacher might be involved within a program should be as culturally relevant as possible. It is important for a teacher to recognize that all strategies do not work for all students and having teachers consider the cultural relevance of an instructional activity may well be a key to helping them to determine how to reach all students (Darling-Hammond, 1997).

6) Although TPCK appears to be a strong foundational framework for restructuring teacher preparation and professional development, it is important for teachers within successful teacher preparation programs to still recognize that not all effective uses of technology are tied directly to content and pedagogy. The effective use of technology is a relatively broad spectrum. In other words, teachers need to be aware that technology can also be an effective element of

classroom management, parent communication, and many other aspects of being a good "teacher."

7) As computers become more ubiquitous and pervasive, technology must still never overshadow the focus on students as individuals, an educational priority that helps to make teachers such a powerful catalyst for positive growth in young people. Teachers of mathematics, like all teachers, must exhibit a "demonstrated caring" that encourages students to take intellectual chances. As suggested by Davis, Maher, and Noddings (1990), "children who feel cared for are more likely to engage freely in the kind of intellectual activity [constructivist dialogue] that we have described here" (pg. 191). When a trusted and caring teacher is leading the class discussions, patiently giving alternate explanations for difficult concepts, or simply helping students to periodically "mess around with mathematical ideas," then these students will indeed be more willing to push their personal limits in understanding mathematics.

A Few Final Thoughts

As institutions who prepare mathematics teachers continue to refine their programs to be more effective in the integration of technology and to more directly address the TPCK of their students, it will be important for these programs to be fully aware of what professional associations like NCTM and AMTE are recommending in this context. Professional associations and coalitions of professional associations, such as the National Technology Leadership Coalition described in the Organizational Structures chapter of this monograph (Chapter 13, Bull, Bell, and Hamonds), are beginning to play a key leadership role in helping to advance an understanding of TPCK as these organizations facilitate collaborative dialogues among professionals. Such collaborative discussions will go a long way toward helping institutions to refine their programs related to technology integration, as we strive to be as effective as possible in preparing teachers for the technological and dynamic world of today.

To help prepare caring, engaging and imaginative mathematics teachers for today's world, it would make sense that those of us in teacher preparation have that goal clearly in mind as we design our programs. Such a goal in our programs is not an easy task and in a final connection to our most imaginative mathematical thinker, it is worthy to consider that Albert Einstein also once said that "every theory should be as simple as possible, but no simpler." In this attempt to look at TPCK in the mathematics classroom and in the preparation of mathematics teachers, I may well have oversimplified the complexity of this important intersection between technology, pedagogy, and content. However, striving for such "simplicity in statement," is itself a time honored approach in the discipline of mathematics, as mathematicians are always striving to state things as simply as possible when we try to understand complex relationships. Simplicity has been a goal of all mathematicians as they define a difficult problem, use symbolic representations, or refine a proof. As an organizing construct for preparing mathematics teachers to use technology effectively in the teaching and learning of their discipline, TPCK would appear to be a worthy construct. It offers additional program focus in helping conceptualize how content, pedagogy, and technology might come together effectively in a teacher preparation program within the context of a mathematics discipline that is itself rapidly evolving due to technology. Mix in a bit of imagination from both the program and the individual, and the teachers graduating from such programs may well have a significant advantage in helping their students to enter the very useful and imaginative realm that has always been mathematics.

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Figure 1: A computer generated Sierpinski Triangle, which is a public domain image from Wikimedia Commons.

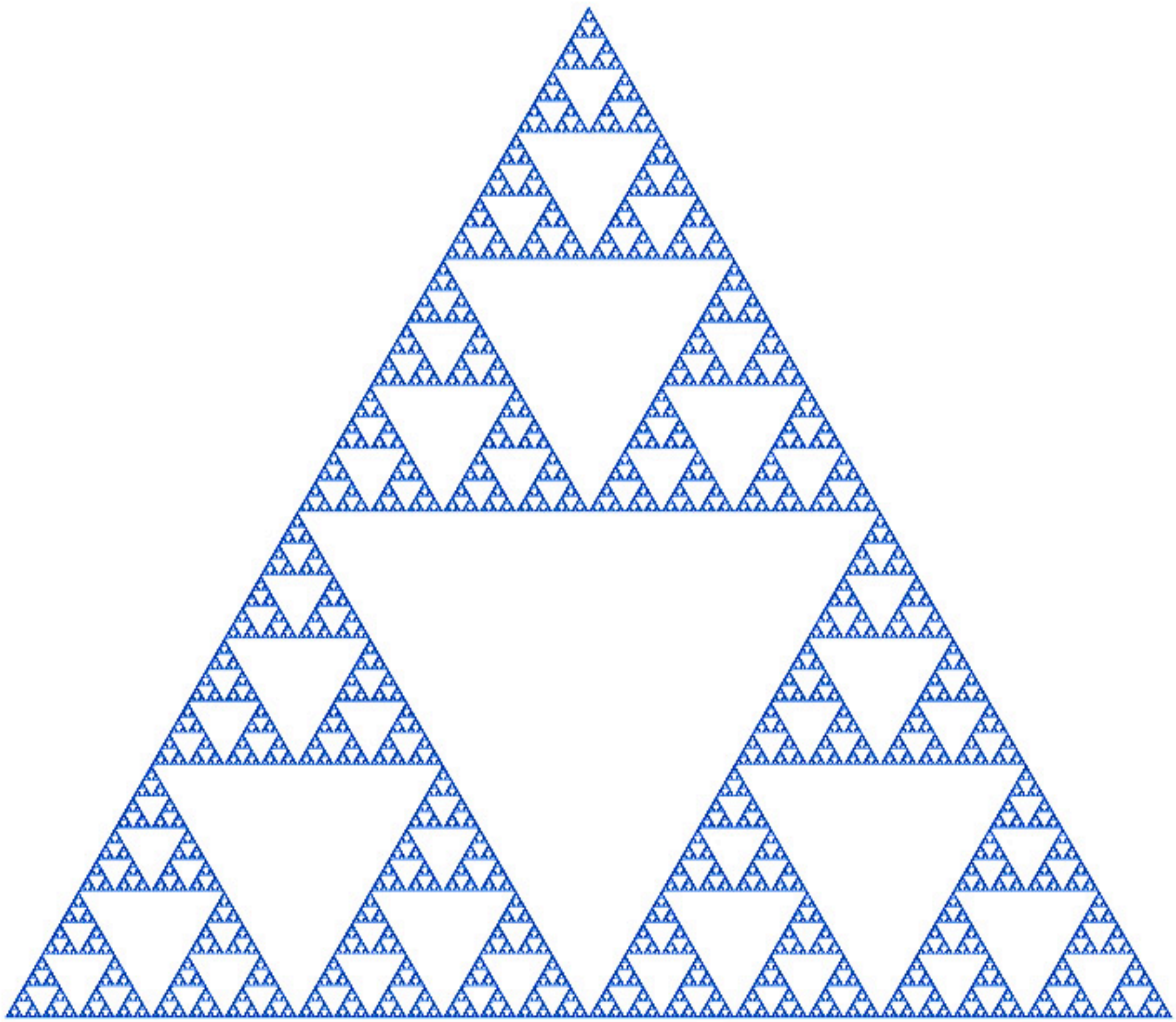

Figure 2: A public domain NASA image from the SeaWiFS satellite of a river system in Norway (visible within snow fields and taken on June 6, 2000).